An analogy strategy for transformation optics

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Abstract

We introduce an analogy strategy to design transformation optical devices. Based on the similarities between field lines in different physical systems, the trajectories of light can be intuitively determined to curve in a gentle manner, and the resulting materials are isotropic and nonmagnetic. Furthermore, the physical meaning of the analogue problems plays a key role in the removal of dielectric singularities. We illustrate this approach by creating two designs of carpet cloak and a collimating lens as representative examples in two- and three-dimensional spaces, respectively. The analogy strategy not only reveals the intimate connections between different physical disciplines, such as optics, fluid mechanics and electrostatics, but also provides a heuristic pathway to designing advanced photonic systems.

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1. Introduction

Transformation optics has aroused interest from a wide spectrum of scientific communities [1–10]. The term was coined based on the fact that Maxwell’s equations are form-invariant under coordinate transformations [2], meaning electromagnetic waves in one coordinate system can be mapped into another one with transformed material properties. This correspondence paves an unprecedented methodology for steering light almost at will [4, 5]. Largely kindled by invisibility cloaks [1, 2, 11], great efforts have been devoted to the design of advanced optical components such as lenses [12–18], waveguides [19–21], light-harvesting devices [22] and polarization converters [23]. Most of these blueprints, however, confront difficulties in realization because of the required complex and extreme material properties after transformation. The state-of-the-art metamaterial toolbox is not yet sufficient to fully construct such devices, especially in the optical regime. The bandwidth is also limited owing to the resonant nature of the building blocks of metamaterials [24, 25]. To effectively facilitate the fabrication, one may need to backtrack to the design procedure. Since the complexity of the media originates from the rough operation of coordinates that gives extraordinary functionality, a judicious selection from the vast possible transformations could result in greatly simplified material parameters. Recent progress has used conformal [1, 18] and quasiconformal mapping [11, 15, 20] to minimize anisotropy for two-dimensional (2D) designs. Unfortunately, the former approach often generates extreme values of material properties, whereas the latter is a numerical technique and is not convenient in many cases for analysis. As a consequence, innovative design approaches are still needed.

Analogy plays an important role in investigating many branches of physics. A famous example is the analogy between lines of force (field lines) and the flow of an incompressible fluid, developed by Maxwell [26]. This intuitive picture is even valid for transformation optics [2, 27]. In many ways, analogies help transfer concepts and ideas from a mature field to a new one. However, aiming at extending the transformation framework to other equations of physics, previous analogy work in transformation optics was mainly concentrated on the similarity of equations [28–33]. Few works have applied the similarity between the field lines of different systems to design and to optimize transformation optical devices.

In this manuscript, we propose a flexible analogy strategy for transformation optical design. Three devices, namely two designs of a 2D carpet cloak and a 3D collimating lens are presented as examples. Light trajectories that visualize space distortion are inspired by the field lines of analogue problems in fluid mechanics and electrostatics, respectively. All these designs are analytical and isotropic. Moreover, extreme values in refractive indices can be avoided by intentionally tuning the configuration of the analogue problems and removing the singularities. These advantages are significant, in comparison with previous works using the conformal or quasiconformal approach. Unlike a recent proposal using abstract spacetime [33], in which the analogue models do not need to have direct physical meaning, our approach takes advantage of the physical picture of the analogue systems. Not necessarily involving complicated mathematical treatment, our ‘physics driven’ analogy strategy provides a shortcut to a variety of optical devices with feasible material parameters.
2. A 2D ‘airfoil’ and ‘Magnus’ carpet cloak

We first discuss the design of carpet cloaks. Similar to an invisibility cloak that can conceal isolated objects as if nothing is present in the space, a carpet cloak hides a bump below it and renders the uneven surface flat to observers. Although some previous work demonstrated that employing anisotropy is a possible route as well [34–36], we focus on isotropic design as most authors have done [11, 37–43]. Taking advantage of the light-fluid analogy, the principle of cloaking can be interpreted with a very intuitive picture, that light is guided to wrap around an object like water flows around a stone [2, 27]. Strictly speaking, in an anisotropic cloak this is not the case, as field lines are bent within the cloak shell in contrast to the streamlines of fluid that are disturbed in the entire space. Nevertheless, the isotropic cloak derived from the Joukowsky transform follows exactly the solution of irrotational flows around a circular cylinder [44].

\[ w = z + \frac{a^2}{z}, \]  

(1)

where \( a \) is the radius of the cylinder, and \( w \) and \( z \) are the 2D coordinates in complex notation before and after the transformation, respectively. Despite being able to perform an omnidirectional cloak by plugging in an additional profile [1], the device without tuning is intrinsically invisible to the incidence from one certain direction, namely the direction that is parallel to the streamlines in the analogue fluid problem. This is true for ray optics but holds for waves only at certain frequencies [45, 46], because the cylinder \( |z| = a \) is mapped into a conducting sheet between \( w = \pm 2a \) along the incidence. However, this topology is just the case for carpet cloaks [11], so there are no constraints for both rays and waves if we locate the sheet on a conducting background and consider only a half-space. Figures 1(a) and (c) show how (1) works, in which the line plots can be interpreted in two ways. From the perspective of fluid mechanics, supposing that an inviscid, incompressible fluid flows uniformly from left to right, the presence of a horizontal wall will not disturb the stream at all. The streamlines (along the horizontal direction) and velocity potential lines (along the vertical direction) in figure 1(a) form a set of orthogonal grids coincident with the Cartesian coordinate frames. When a semi-cylinder is immersed by the bottom wall, fluid is displaced and forced to flow tightly around the bump, whereas the potential lines terminate at the reshaped wall and remain orthogonal to the streamlines. On the other hand, from a transformation optics point of view, replacing the streamlines, potential lines and wall respectively with light rays, wave fronts and a reflective mirror, the chart is still correct. The only difference is, the bending of field lines should now be attributed to curved space, or equivalently, the transformation medium.

The above analogy applies to incidence in merely one direction for the uniform flow, while light can go further because of its wave nature. In figure 1(a), one can draw at will oblique lines that are folded symmetrically to represent the reflection of light by a flat mirror, and their images in figure 1(c) reconstruct the diagram of reflection in the transformed space [47].

Obviously, there are two issues concerning this semi-cylinder carpet cloak. First, close to the bump, the reflected light is shifted a large distance from its pre-image in the virtual space. For instance, the cyan curves in figure 1(c), which are expected to be adjacent to the reference ray shown in black as in figure 1(a), deviate rightward for almost one grid of the background frames. This means the bump is still detectable by measuring the lateral shift of the reflected rays [48]. The shift diminishes with the increasing size of the cloak, as it comes merely from the...
boundary truncation at a finite distance, but does exist for any practical isotropic cloaks. The second problem is more critical. In figure 1(d), three distinct regions in the profile are noticeable. The top area with an index near 2 is attainable though challenging at visible frequencies, while two singularities at the bottom and the surrounding low-index areas are indeed hard to realize. To tackle these problems, we revisit the light-fluid analogy from another angle. In transformation optics, the refractive index under conformal mapping follows \( n = \frac{\mathrm{d}w}{\mathrm{d}z} \cdot n' \), where \( n' \) equals 1 for an empty virtual space. The same formula is valid as well in fluid mechanics, where the magnitude of the velocity is given by \( v = \frac{\mathrm{d}w}{\mathrm{d}z} \). Although their physical

\[ \begin{align*}
\text{Figure 1.} & \text{ Left column: ray traces of light in a vacuum (a) and two transformation media (c) and (e). The frames in blue show the original and transformed coordinate lines. The horizontal black lines at the bottom represent the ground plane, and the red curves in (c) and (e) correspond to the contours of the bumps. The V-shaped black lines are drawn as a reference to show the trajectory of light with an incident angle of 45° in free space. The solid-dashed line pairs in color show bundles of light with incident angles of 15° (yellow), 45° (cyan) and 75° (purple) from the left, respectively. (a) A half-space bounded by a perfect conductor. (b) A Gaussian beam launched at 45° is simply reflected at the same angle. (c) A semi-cylinder carpet cloak and (d) its refractive index profile. Singularities appear at sharp corners. (e) A semi-airfoil carpet cloak and (f) its index profile. The singularity on the right is avoided by the smooth transition of the trailing edge. If we treat the blue frames in the left column as the streamlines (along the horizontal direction) and velocity potential lines (along the vertical direction) of fluids, then (d) and (f) are maps of fluid velocity, while the singularities are the points where the velocity vanishes.}
\end{align*} \]
meanings are totally different, we can manipulate the optical media with inspiration from the well-studied flow problems for an ideal fluid.

Now we are able to predict the index profile with known solutions of fluid mechanics, as in our consideration that the refractive index is analogous to the magnitude of the flow velocity. When flowing around a semi-cylinder, the bottom-most streamline has no way to veer but has to stop at sharp corners, forming two stagnation points therein. The adjacent flow, represented by the green curve in figure 1(c), is also slowed down sharply near the corners, queuing to bypass the bump. However, to be finally restored to a uniform flow in the far-field, the stream will speed up on the top region, compensating for the previous delays. This explains the index distribution in figure 1(d) in the language of fluid mechanics, which might not be deduced so straightforwardly from the picture of light itself. The removal of the singularity and alleviation of field distortion become easy in the analogue fluid system. A simple test can be done by performing an inverse Joukowsky transform on (1), that is,

\[ z' = \frac{w' \pm \sqrt{w'^2 - a'^2}}{2}, \]

where \( z' \) and \( w' \) are coordinates before and after the transformation, and \( a' \) is a constant smaller than \( a \). Equation (2) further maps the semi-cylinder into a semi-airfoil with zero angle of attack. Figures 1(e) and (f) show, respectively, the improvement on the lateral shift and the dependence of the material profile on the geometry. According to the Kutta condition, the flow velocity must be zero at sharp corners while keeping continuous on the smooth parts. This contrast is clearly reflected at the leading and trailing edge.

The final step to construct a carpet cloak is to make both edges smooth to remove any singularity. Noting that the airfoil is relatively flat on top, coordinate lines at the apex are lifted up with only slight lateral perturbation. We thus simply perform a reflection by flipping over the rear of the airfoil about the dashed axis in figure 2(a). This operation can be in turn mapped to the virtual space in figure 1(a), ensuring the carpet is optically equivalent to a conducting plane. We examined the effectiveness of the design with several challenging tests\(^7\), where the bump height is almost twice the wavelength. In all simulations, the cloak works well in keeping the shape of the reflected beams as in ray tracing. By comparing figures 1(b) and 2(d), one may find a lateral shift still exists. Indeed, this is an intrinsic problem of isotropic cloaks with a truncated boundary \([48]\). However, the shifting effect is not significant at long wavelengths \([47]\) and can be reduced by lowering the device physically \([40]\) or optically \([49]\). Further simulations even confirmed that the performance will not be affected much if the present profile in figure 2(b) is truncated by a reasonably lower height.

Despite different governing equations, the intuitive analogy between fluid and light, or specifically, between streamlines and rays, enables one to hold a prior picture in mind when constructing a device. In fluid mechanics, the general motion of fluid can be thought of as the superposition of some basic flows, each of which is described by a simple analytic function. For instance, a sink is given by \( w = -\log(z) \) and a vortex by \( w = \pm i \cdot \log(z) \). Noting that their derivatives, \( \frac{dw}{dz} = 1/|z| \), are the same, it would be interesting to think about how light behaves in a medium with a refractive index determined by such maps. In fact, because a sink absorbs fluid isotropically, light rays carrying only radial momentum are attracted straight towards the central singularity in the analogue medium. Similarly, light with purely tangential momentum

\(^7\) See the supplementary data for more simulation results (available at stacks.iop.org/NJP/16/063008/mmedia).
runs in circles as fluid in a vortex does. Then it is not surprising to imagine that an arbitrary incidence should follow both trends but finally drop into the center, exhibiting a spiral trace as shown in figure 3(a). This analogy provides a very simple way to derive an optical black hole, without employing either the Hamiltonian [50] or the Lagrangian [51]. Systems containing sinks, such as doublet (dipole, $w=1/z$), quadrupole ($w=1/z^2$) and Rankine half-body ($w = z - \log (z)$), can be used to create different profiles for absorbing light [52, 53]. Although the field line patterns may become abstract with multipoles, the key point is that all streamlines terminate at a finite area where the sinks locate.

Furthermore, in analogy to fluid mechanics, the functions describing basic flows can be organized as mathematical building blocks to create complex ray patterns for optical applications. Based on the knowledge obtained from analogue problems, one can choose proper functions on purpose to optimize the resulting materials. Here, we introduce another design of a carpet cloak analogous to an object in flow with circulation. Recalling figures 1(c) and (d), one may suggest that choosing a streamline above the ground as the inner boundary of the cloak is also a method of singularity removal. In principle this works, but drawbacks are evident. As seen in figure 1(d), to tailor the profile to a feasible range, the streamline should be chosen quite a distance from the current ground, leading to long trailing edges and an almost flat contour. On the other hand, the outer boundary is unaltered, meaning that coordinate distortion is not reduced, even if the bump is quite smooth. Inspired by the Magnus effect in fluid mechanics, we add a vortex term into (1) and get
\[ w = z + \frac{a^2}{z} + i\Gamma \cdot \log(z), \]  

where \( \Gamma \) is the strength of the vortex. With the cylinder rotating clockwise, the fluid at the original stagnation points receives a velocity and flows around the object. However, new stagnation points arise somewhere on the lower half of the cylinder, as the motion of the original flow and the vortex cancel out. The pattern of streamlines is illustrated in figure 3(b), which shows that streamlines in the upper-half plane now intersect with the ground and therefore shorten the undesired edges. Equivalently, the dielectric singularities are depressed below the ground, outside the region of our concern. It should be noted that the index at the top of the cylinder is larger than before (figure 1(d)), because the flow velocity in the upper-half plane is accelerated by the vortex. However, the payoff from truncating singularities at the bottom is even greater, resulting in a more feasible index range in total.

In order to mitigate the coordinate distortion to the largest extent, we simulate an elliptical ‘Magnus’ carpet cloak in figures 3(c) and (d), which is obtained by transforming the circular cylinder in figure 3(b) into an ellipse. This operation is conformal as well and is commonly used in fluid mechanics dealing with geometries other than circles. Despite the profile range being larger than in the ‘airfoil’ case (the cost of fitting a more isolated bump), the performance is well...
kept. Moreover, one can easily choose another conformal streamline above the current inner boundary and achieve both a larger cloaking area and a smaller index range.

3. 3D ‘charge’ collimating lens

Since the only basis of our analogy strategy is the similarity between field lines, there are no limits on the dimensions of the systems involved. Under some circumstances, it allows us to go beyond conformal mapping for isotropic devices in three dimensions [54]. As another example, we design a collimator in 3D space, which consists of a point source and a lens with a flat aperture normal to the direction of emission [16–18, 55, 56].

A point source of light has many counterparts generating flux in other branches of physics. To attain field lines orthogonal to a certain surface, the most intuitive analogue might be the electrostatic fields suspended at a perfect conductor. The idea of drawing parallels between optics and electrostatics is not new but normally one-way from optics to electrostatics. For instance, an electrostatic lens that controls electron beams is analogous to an optical lens dealing with light. Here, we employ this analogy in the inverse way. When setting a charge near a conducting plane, the distribution of electric field lines between the charge and the conductor mimics the light rays inside a collimating lens. Note that the same analogue picture can be drawn from steady heat transfer, between a point heat source and an isothermal wall. However, different to the carpet cloak example where the region under consideration is source free, here we cannot simply assign equipotential surfaces to wave fronts owing to the presence of the point source. Comparing the eikonal equations in both systems, that is, $n = |\nabla S|$ for ray optics, where $n$ and $S$ are functions describing the refractive index and wave fronts, and $E = |\nabla \Phi|$ for electrostatics, with $E$ and $\Phi$ representing the electric field strength and potential, we realize that a direct map of potential to phase will lead to a singularity of refractive index at the source. To avoid this trouble while maintaining the analogy between light rays and electric field lines, we redefine the function of wave fronts with a scaling transformation [27]

$$S = \arctan \left( \frac{1}{\Phi} \right).$$

In this way, $S$ equals zero at the source and accumulates to a finite value at the aperture plane, thus satisfying the physical meaning of phase. Figure 4(a) illustrates the resulting index profile in cross section. As can be seen, its value at the apex is adjusted to unity without introducing new singularities. The boundary contour, which coincides with the trace of an analogue field line as well, is chosen to be parallel to the aperture at the apex. Nevertheless, one can choose different contours truncating the profile to avoid the areas where indices are less than 1. The simulation of 3D collimating is somewhat difficult, mainly because of the lack of a source with isotropic radiation [57, 58]. Here we perform the test inversely by illuminating the lens along its axis with a Gaussian beam. Shown in figure 4(b), the beam converges precisely on the apex as designed and the flat wave fronts are well preserved, despite a slight reflection caused by impedance mismatch at the surface. It is straightforward to deduce, from both the symmetry of the field line patterns in the analogue problem and the cross-sectional field plots in figure 4(b), that the index profile is also valid for 2D geometry, where the structure is translationally invariant along the out-of-plane axis. We simulated this case by setting a line current at the apex exciting a TE polarized cylindrical wave, and a perfect conductor was added on the back acting as a mirror or reflector. The electric field pattern in figure 4(c) clearly shows
the conversion of wave fronts to a highly collimated beam, and, specifically, the power flow lines follow exactly the traces of electric field lines emanating from a point charge. Depending on the choice of $\Phi$ in (4), the index distribution is not unique. We note that Benítez and coworkers discussed a similar profile with a smaller lower bound starting from the scalar Helmholtz equation and the field of a dipole [59]. Another family of 2D dipole lenses was also reported there. However, the derivation of the latter does not necessarily need to use an eikonal equation and scaling transformation. For lenses employing light rays analogous to the field lines of a 2D charge or dipole, the same profiles can be derived through conformal mapping. In fact, similar to the generalization of Maxwell’s fish eye to perfect optical instruments [60], each index profile of radial symmetry in the 2D space corresponds to a solution of the 2D dipole lenses with the connection of a Möbius transformation [56]. This procedure is not available in the 3D space but can fortunately be superseded by transforming the eikonal equations.

While a pair of charges opposite in sign illustrate the field lines for lensing, a pair with the same sign perform differently, giving rise to the chance to eliminate the interference of waves [61, 62]. In [62], the result designed from the geometrical similarity between wave fronts and equipotential surfaces even satisfies the exact equations of wave optics. This means our analogy strategy may have deeper connections with other analogy works based on the formal similarity of governing equations.

Figure 4. (a) Cross-sectional view of the index profile of the 3D collimating lens, whose thickness and aperture size are 1 and 3.5 in normalized units, respectively. (b) An $x$-polarized Gaussian beam incident from bottom to top is focused on the apex. The wavelength is 0.25. Two vertical slices show, without loss of generality, the distribution of the electric field $E_x$ within the $x$–$z$ plane (the in-lens part only) and the plane making an angle of 84° with respect to the $x$–$z$ plane, respectively. Different color maps are used for clarity. The bottom slice shows the power distribution of the total field. (c) In the 2D version, a cylindrical wave radiated from a line current at the apex is converted into a collimated beam, with power flow lines (in red) inside the lens coinciding with the traces of electric field lines emanating from a point charge. The wavelength is 0.12.
4. Conclusion

In conclusion, we have shown the use of an analogy strategy for transformation optical design. Based on the characteristics of functionality, three devices are created intuitively by constructing analogue problems with similar field line patterns but in a different physical system. Material singularities can be removed by referencing the physical meaning of their analogues and intentionally manipulating the geometry of the analogue problem. For devices performing a unidirectional behavior, the analogy idea bridges the transformation approach and ray optics formulae to create feasible 3D profiles, which are hardly achievable using either technique alone. Although the examples shown here are isotropic, some of which can be obtained as well by solving Laplace’s equation [38, 63], we believe it is possible to find analogies to anisotropic systems for biaxial [64] or more advanced devices [65]. Meanwhile, numerical techniques should not be excluded for complex geometries. The analogy strategy provides a heuristic way to design and optimize transformation media. Vast applications are expected, promised by the fact that there are many other physical systems exhibiting similar field line patterns among different disciplines.

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